TEMA 5: EDP DE 2º ORDEN

EDP lineal homogénea con coeficiente constantes de 2º orden.

$$a z_{xx} + h z_{xy} + c z_{yy} + l \quad x + m z_y + z = 0$$

$a, b, c, l, m \in \mathbb{R}$

z? $z(x,y)$

Supondremos $b = 0$.

En este caso, asumiendo

$z(x,y) = X(x) \cdot Y(y)$ (separación de variables)

Podremos encontrar soluciones.

Ecuación del calor:

Buscar una función $u(x,t)$

- $K \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ ($k>0$, difusividad térmica)
- $u(0,t) = 0$, $u(L,t) = 0$ para toda t.
- $u(x,t) = f(x)$, con $f(x)$ dada ($0 < x < L$)

$\boxed{u(x,t)}$

$f(x)$ es la temperatura siempre $t=0$ de la barra.

Busquemos las soluciones.

suponganmos que $u(x,t) = X(x)T(t) \rightarrow$

$u_{xx} = (X'T)'$

$$\frac{\partial u}{\partial x} = X'T + X'T_x = X'T$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = X'T'$$

$\rightarrow KX''T = X'T' \quad \begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$

$u(0,t) = 0 = X(0)\,T(t)$

$u(L,t) = 0 = X(L)\,T(t)$

$$\frac{X''}{X} = \frac{T'}{KT} = \lambda \in \mathbb{R} \rightarrow \left| \begin{array}{l} \dfrac{X''}{X} = \lambda \\ \dfrac{T'}{KT} = \lambda \end{array} \right. \rightarrow \left| \begin{array}{l} X'' - \lambda X = 0 \\ T' - \lambda K T = 0 \end{array} \right.$$

$$\begin{cases} r^2 - \lambda = 0 \\ r - \lambda K = 0 \end{cases}$$

- Suponemos que $\lambda = -c^2$ con $c > 0$.

$X'' - \lambda X = 0 \qquad r^2 = -c^2 \quad ; \quad r = \pm ic$

$B = \{\operatorname{sen} cx, \cos cx\}$

$X(x) = c_1 \operatorname{sen}(cx) + c_2 \cos(cx) \longrightarrow$

$X(0) = 0 = c_2$

$X(L) = 0 = c_1 \operatorname{sen}(cL) \rightarrow \operatorname{sen}(c.L)$

$c.L = n\pi$ con $n \in \mathbb{Z}$

$\rightarrow X(x) = c_1 \operatorname{sen}\left(\frac{n\pi}{L}\right)$

$T' - \lambda K T = 0$; $r = \lambda k = -c^2 k = -\frac{n^2 \pi^2}{L^2} k$

$B = \{ e^{-\frac{n^2\pi^2 k}{L^2} t} \}$

$$T(t) = C_3 e^{-\frac{n^2 \pi^2 k}{L^2} \cdot t}$$

Suponemos que $u(x,t) = X(x)\,T(t)$

$\rightarrow u(x,t) = c_1 \operatorname{sen}\left(\frac{n\pi}{L} x\right) c_3\, e^{\left(-\frac{n^2\pi^2 k}{L^2} t\right)}$

- Suponemos que $\lambda = 0$

$X'' - \lambda X = 0$; se reduce a $X''(x) = 0$

así $X(x) = ax + b$

$X(0) = 0 \rightarrow b = 0$

$X(L) = 0 \rightarrow aL = 0 \rightarrow a = 0 \rightarrow X(x) = 0$

Así que $u(x,t) = 0$, que es la solución trivial.

- Suponemos que $\lambda = c^2$ $(c > 0)$

$u(x,t) = A_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 k}{L^2} t}$

T5.2

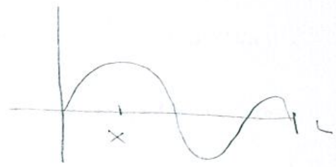
Ecuación de Ondas.

Buscar $u(x,t)$

$$a^2 \frac{\partial u^2}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (0 < x < t) \qquad (t > 0)$$

$$u(0,t) = 0 \quad ; \quad u(L,t) = 0 \qquad (t > 0)$$

$$u(x,0) = f(x) \quad ; \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$



Método de separación de Variables

$$u(x,t) = h(x)\, k(t)$$

$$a^2 h''(x)\, k(t) = k''(t)\, h(x)$$

$$\frac{h''(x)}{h(x)} = \frac{k''(t)}{a^2 k(t)}$$

$$\left.\begin{array}{l} \dfrac{h''(x)}{h(x)} = \lambda \\[2ex] \dfrac{k''(t)}{a^2 k(t)} = \lambda \end{array}\right\} \Rightarrow \left.\begin{array}{l} h''(x) - \lambda h(x) = 0 \\[2ex] k''(t) - \lambda a^2 k(t) = 0 \end{array}\right| \begin{array}{l} (r^2 = \lambda) \\[2ex] (r^2 = \lambda a^2) \end{array}$$

si $\lambda > 0$

$$h(x) = c_1 e^{\sqrt{\lambda}\, x} + c_2 e^{-\sqrt{\lambda}\, x}$$

$$k(t) = c_3 e^{-a\sqrt{\lambda}\, t} + c_4 e^{-a\sqrt{\lambda}\, t}$$

$u(0,t) = 0 = h(0)\, k(t)$ para todo t

* Buscamos soluciones que no sean la trivial.

$$h(0) = c_1 + c_2$$

$$h(L) = 0 = c_1 e^{\sqrt{\lambda}\, t} + c_2 e^{-\sqrt{\lambda}\, t}$$

$$u(L,t) = 0 = h(L)\, k(t)$$

$k = 0 \to$ no puede ser

si $\lambda = 0$

$$h''(x) = 0$$

$$h(x) = p(x) + q$$

$$h(0) = 0 = h(L)$$

$$h(0) = 0 \longrightarrow q = 0 \qquad h(x) = px$$

$$h(L) = 0 = pL$$

$h = 0 \rightarrow$ no puede ser

si $\lambda < 0$

$$h(x) = c_1 \operatorname{sen} \sqrt{-\lambda}\, x + c_2 \cos \sqrt{-\lambda}\, x$$

$$(r = \pm i\sqrt{-\lambda})$$

$$r^2 = \lambda a^2 \rightarrow \pm i\sqrt{-\lambda a^2} = \pm i a \sqrt{-\lambda}$$

entonces:

$$h(0) = 0 \ ; \ h(L) = 0$$

$$h(0) = c_2 = 0$$

$$h(L) = 0 = c_1 \operatorname{sen} \sqrt{-\lambda} \rightarrow \operatorname{sen}(\underbrace{+\sqrt{-\lambda}\, L}_{n\pi,\ n \in \mathbb{Z}}) = 0$$

$$\sqrt{-\lambda}\, L = n\pi$$

$$\sqrt{-\lambda} = \frac{n\pi}{L}$$

$$h(x) = c_1 \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

Entonces:

$$h(x) = c_1 \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$k(t) = c_3 \operatorname{sen}\left(\frac{a n\pi}{L} t\right) + c_4 \cos\left(\frac{a n\pi}{L} t\right)$$

$$u(x,t) = c_1 \operatorname{sen}\left(\frac{n\pi x}{L}\right)\left[c_3 \operatorname{sen}\left(\frac{a n\pi}{L}\right) t + c_4 \cos\left(\frac{a n \pi}{L} t\right)\right]$$

solución:

$$u(x,t)=\left[A_n \operatorname{sen}\left(\frac{a n\pi}{L}t\right)+B_n \cos\left(\frac{an\pi}{L}t\right)\right]\operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

$$u(x,0)=f(x) \Rightarrow u(x,0)=\sum {}_n \operatorname{sen}\left(\frac{n\pi x}{L}\right)=f(x)$$

$$u(x,t)=\sum_{n=0}^{\infty}\left[B_n \operatorname{sen}\left[\frac{an\pi}{L}t\right]+A_n\left[\frac{an\pi}{L}t\right]\right]\operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

→ "Los A_n son los coeficientes de Fourier de $f(x)$ extendida a $[-L,L]$ como función impar".

$$A_n=\frac{2}{L}\int_0^L f(x)\operatorname{sen}\left(\frac{n\pi x}{L}\right)dx \quad \Bigg\| \quad \text{si } L=\pi \quad A_n=\frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{sen}(nx)\,dx$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x) \Rightarrow \sum_{n=0}^{+\infty}\operatorname{sen}\left(\frac{n\pi x}{L}\right)\left[B_n\cos\left(\frac{an\pi}{L}t\right)\frac{an\pi}{L}\right]=g(x)$$

$$\sum_{n=0}^{+\infty}\frac{an\pi}{L}B_n\operatorname{sen}\left(\frac{n\pi x}{L}\right)=g(x)$$

→ " $\frac{an\pi}{L}B_n$ son los coef. de Fourier de la extendida de $g(x)$ a $[-L,L]$".

$$B_n=\frac{L}{an\pi}\int_0^L f(x)\operatorname{sen}\left(\frac{n\pi x}{L}\right)dx \quad \Bigg\| \quad \text{si } L=\pi \quad B_n=\frac{1}{an}\int_0^{\pi} f(x)\operatorname{sen}(nx)\,dx$$

continuación de la Ec. del calor:

$$u(x,t) = A_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 k}{L^2} t}$$

$$u(x,t) = \sum_{n=1}^{+\infty} A_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 k}{L^2} t}$$

Si queremos $u(x,0) = f(x)$

$$\sum A_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) = f(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x) \operatorname{sen}\left(\frac{n\pi x}{L}\right) dx$$

Problema 2.

ii) $L = \pi$

$$f(x) \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$\boxed{u(x,t) = \sum_{n=1}^{+\infty} A_n \operatorname{sen}(nx)\, e^{-n^2 kt}}$$

Ec. del flujo del calor

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \operatorname{sen}(nx)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \operatorname{sen}(nx) dx =$$

$$= \frac{-2}{n\pi}\left[\cos(nx)\right]_0^{\pi/2} = \frac{-2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos 0\right]$$

$$\cos\frac{n\pi}{2} \begin{cases} 0 \text{ si } n \text{ impar} \\ 1 \text{ si es multiplo de } 4 \\ -1 \text{ si es par pero no multiplo de } 4. \end{cases}$$

$$= \begin{cases} \frac{2}{n\pi} \text{, si } n \text{ es impar} \\ 0 \text{, si } n \text{ es } 4 \rightarrow \text{multiplo de } 4. \\ \frac{4}{n\pi} \text{, si } n = 2, 6, 10 \ldots \end{cases}$$

15.4

Problema 2.

i) $$K \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (0<x<\pi,\ t>0)$$

$$u(0,t)=0, \quad u(\pi,t)=0 \qquad t>0$$

$$u(x,0) = \operatorname{sen} x + 2\operatorname{sen}(3x)$$

cuando f(x) y g(x) no sean n°s (const) se aplica

$$u(x,0) = \operatorname{sen} x + 2\operatorname{sen}(3x) = f(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \operatorname{sen}(nx)\, e^{-n^2 \pi k}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{sen}(nx)\,dx \qquad \left[\sum_{n=1}^{\infty} A_n \operatorname{sen}(nx) = f(x)\right]$$

↳ nos fijamos en la fórmula: $\operatorname{sen} x + 2\operatorname{sen} 3x$

$A_1 = 1 \qquad A_3 = 2$

$A_n = 1 \qquad A_3 = 2$ → es n

$$u(x,t) = \operatorname{sen} x\, e^{-tk} + 2\operatorname{sen} 3x\, e^{-9tk}$$

n^2

Problema 3.

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (0<x<\pi,\ t>0)$$

$$u(0,t)=0, \quad u(\pi,t)=0 \quad (t>0)$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= g(x) \end{aligned}\right\} \ (0<x<\pi$$

$$u(x,t) = \sum_{n=1}^{+\infty} \left[A_n \cos(nat) + B_n \operatorname{sen}(nat)\right] \operatorname{sen}(nx)$$

$$\sum_{n=1}^{\infty} A_n \operatorname{sen}(nx) = f(x) \xrightarrow{\text{"Fourier"}} A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{sen}(nx)\,dx$$

$$\sum_{n=1}^{\infty} B_n\, na \operatorname{sen}(nx) = g(x) \rightarrow B_n\, na = \frac{2}{\pi}\int_0^{\pi} g(x)\operatorname{sen}(nx)\,dx$$

Problema 4.

i) $f(x)=0$; $g(x) = \text{sen } x - 3\,\text{sen } 4x$

$\downarrow$ $A_n = 0$ para todo $n \geq 1$

$\downarrow$ $B_1\, 1 \cdot a = 1 \rightarrow B_1 = \frac{1}{a}$

$B_4\, 4 \cdot a = -3 \rightarrow B_4 = \frac{-3}{4a}$

$$u(x,t) = \frac{1}{a}\,\text{sen}(at)\,\text{sen } x - \frac{3}{4a}\,\text{sen}(4ta)\,\text{sen}(4x)$$

ii) $f(x) = 0$; $g(x) = \pi x$

$\downarrow$ $A_n = 0 \; \forall n \geq 1$

$\downarrow$ Tenemos que hacerlo con la fórmula.

$$B_n = \frac{2}{n\cancel{\pi}a}\int_0^\pi \cancel{\pi} x\,\text{sen}(nx)\,dx$$

$$= \frac{2}{na}\int_0^\pi x\,\text{sen}(nx)\,dx =$$

$$\underset{\text{por partes}}{=} \frac{1}{n}\left[x\cos(n\pi)\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\,dx =$$

$$= \frac{-\pi}{n}(-1)^n + \cancel{\frac{1}{n^2}(\text{sen}(n\pi))}^{\,0} = \frac{(-1)^{n+1}\pi}{n}$$

$$B_n = \frac{2}{na}\;\frac{(-1)^{n+1}\pi}{n} = \frac{2\pi\,(-1)^{n+1}}{n^2 a}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2\pi(-1)^{n+1}}{n^2 a}\,\text{sen}(nat)\,\text{sen}(nx)$$

i)
$$f(x)=\begin{cases} x & 0<x<\frac{\pi}{2} \\ \pi-x & \frac{\pi}{2}<x<\pi \end{cases} \quad ; \quad g(x)=0 \to B_n=0$$

$$A_n=\frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{sen}(nx)\,dx$$

$$=\frac{2}{\pi}\left[\int_0^{\pi/2} x\,\mathrm{sen}\,nx\,dx+\int_{\pi/2}^{\pi}(\pi-x)\,\mathrm{sen}\,nx\,dx\right.$$

$$\int x\,\mathrm{sen}(nx)\,dx=\frac{-x}{n}\cos(nx)+\frac{1}{n}\int\cos(nx)\,dx=$$

$$=\frac{-x}{n}\cos(nx)+\frac{1}{n^2}\,\mathrm{sen}\,nx$$

$$\cancel{A_n=\frac{2}{\pi}\left[\frac{-x}{2n}\cos nx+\frac{1}{n^2}\,\mathrm{sen}\,nx\right]_0^{\pi/2}+\ldots}$$

$$A_n=\frac{2}{\pi}\left[\left[\cancel{\frac{-\pi}{2n}\cos\left(\frac{n\pi}{2}\right)}+\frac{1}{n^2}\,\mathrm{sen}\left(\frac{n\pi}{2}\right)\right]+\right.$$

$$+\pi\left[-\frac{\cos(nx)}{n}\right]_{\pi/2}^{\pi}-\left[-\cancel{\frac{\pi}{n}\cos(n\pi)}+\cancel{\frac{1}{n^2}\,\mathrm{sen}(n\pi)}\right.$$

$$\left.\left.+\frac{\pi}{2n}\cos(n\pi/2)-\frac{1}{n^2}\,\mathrm{sen}\left(\frac{n\pi}{2}\right)\right]\right]$$

$$A_n=\frac{2}{\pi}\left[\frac{2}{n^2}\,\mathrm{sen}\left(\frac{n\pi}{2}\right)-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$A_n=\begin{cases}\frac{-1}{n}\cos\left(\frac{n\pi}{2}\right) & \text{si } n \text{ es par}\\ \frac{4}{\pi n^2}\,\mathrm{sen}\left(\frac{n\pi}{2}\right) & \text{si } n \text{ es impar}\end{cases}$$

$$n \text{ es par} \begin{cases} -\frac{1}{n} & \text{si } n=4 \\ \frac{1}{n} & \text{si } n\neq 4\end{cases}$$

$$n \text{ es impar} \begin{cases} \frac{4}{\pi n^2} & \text{si } n=4r+1 \\ \frac{-4}{n^2\pi} & \text{si } n=4r+3\end{cases}$$